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The Partition Dimension of a Subdivision of a Complete Graph

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Abstract

The concept of a graph partition dimension was introduced by Chartrand *et al*. (1998). Let $\Pi = \{L_1, L_2, L_3, \cdots, L_k\}$ be a k-partition of $V(G)$. The *representation* $r(v|\Pi)$ of a vertex v with respect to Π is the vector $(d(v, L_1), d(v, L_2), \cdots, d(v, L_k))$. The partition Π is called a *resolving partition* of G if $r(w|\Pi) \neq r(v|\Pi)$ for all distinct $w, v \in V(G)$. The *partition dimension* of a graph, denoted by $pd(G)$, is the cardinality of a minimum resolving partition of G.

This paper considers in finding partition dimensions of graphs obtained from a subdivision operation. In particular, we derive an upper bound of partition dimension of a subdivision of a complete graph K_n with $n \geq 9$. Additionally for $n \in [2, 8]$, we obtain the exact values of the partition dimensions.

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1. Introduction

Let $G = (V, E)$ be a connected graph. The *distance* $d(u, v)$ from a vertex u to a vertex v is defined as the length of a shortest path between u and v. Let $L = \{v_1, v_2, \cdots, v_k\}$ be a subset of $V(G)$, the *distance* $d(v, L)$ from a vertex v to the set L is $\min\{d(v, v_i)|v_i \in L\}$. Let $\Pi = \{L_1, L_2, L_3, \cdots, L_k\}$ be a k-partition of $V(G)$. The *representation* $r(v|\Pi)$ of a vertex v with respect to Π is the vector $(d(v, L_1), d(v, L_2), \cdots, d(v, L_k))$. The partition Π is called a *resolving partition* of G if $r(w|\Pi) \neq r(v|\Pi)$ for all distinct $w, v \in V(G)$. The *partition dimension* of a graph, denoted by $pd(G)$, is the cardinality of a minimum resolving partition of G. A vertex v is said to be a *dominant vertex* if $d(v, L_i) \leq 1$ for each $i \in [1, k]$.

Let G be a graph on n vertices with the vertex-set $V(G)$. The *subdivision graph* $S(G)$ of a graph G is the graph obtained from G by replacing each edge uv of G by a new vertex w and the two new edges uw and vw [4]. The vertex w is called a *subdivision vertex* on uv. For any graph G, the subdivision of graph G will always be bipartite, since the vertex-set can be partitioned into V_1 and V_2 where $V_1 = V(G)$ and V_2 is the set of all subdivision vertices, with any edge in G connects one vertex in V_1 and one vertex in V_2. Therefore, the partition dimension of a subdivsion of a

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graph is bounded above by the the bounds for bipartite graphs as follows.

Theorem 1. [3] *Let G be a bipartite graph with partite set V_1 and V_2, then*

1. $pd(G) \leq |V_1| + 1$, *if* $|V_1| = |V_2|$, *and*
2. $pd(G) \leq max\{|V_1|, |V_2|\}$, *if* $|V_1| \neq |V_2|$.

In this paper, we derive an upper bound for the partition dimension of the subdivision of a complete graph $S(K_n)$. The upper bound of the partition dimension of $S(K_n)$ is an improvement to the bound given in Theorem 1.

2. Main Results

From now on, let $V(K_n) = \{v_1, v_2, \cdots, v_n\}$. The vertex-set of $S(K_n)$ is $V(S(K_n)) = \{v_1, v_2, \cdots, v_n\} \cup \{x_{i,j}|i, j \in [1, n],\ \ i < j\}$. Note that $x_{i,j}$ are the subdivision vertices on v_iv_j. The edge-set of $S(K_n)$ is $E(S(K_n)) = \{v_ix_{i,j}|i, j \in [1, n]\ and\ i < j\} \cup \{v_jx_{j,i}|i, j \in [1, n]\ and\ i < j\}$.

We will find the partition dimension of $S(K_n)$ for $n \in [2, 8]$ which will be presented in Theorem 15. To do so, the following lemmas are needed.

Lemma 2. *Let $n \geq 5$, $p \geq 3$, and $\Pi = \{L_1, L_2, \cdots, L_p\}$ be a resolving partition of $S(K_n)$. Then,*
(i) $d(v_i, L_k) \leq 3$ for all $k \in [1, p]$ and $i \in [1, n]$.
(ii) $d(x_{i,j}, L_k) \leq 4$ for all $k \in [1, p]$ and $i, j \in [1, n]$.

Proof. Since $d(v_i, v_j) \leq 2$ for $i, j \in [1, n]$ and $d(v_i, x_{j,k}) \leq d(v_i, v_j) + d(v_j, x_{j,k})$, we obtain $d(v_i, x_{j,k}) \leq 2 + 1 = 3$ for $i, j, k \in [1, n]$. This implies $d(v_i, L_k) \leq 3$ for each $k \in [1, p]$ and $i \in [1, n]$.

Now, because of $d(x_{i,j}, x_{s,t}) \leq d(x_{i,j}, v_j) + d(v_j, v_s) + d(v_s, x_{s,t})$ for $i, j, s, t \in [1, n]$, we get $d(x_{i,j}, x_{s,t}) \leq 1 + 2 + 1 = 4$. Hence, we obtain $d(x_{i,j}, L_k) \leq 4$ for each $k \in [1, p]$. □

Lemma 3. *Let $n \geq 5$, $p \geq 3$, and $\Pi = \{L_1, L_2, \cdots, L_p\}$ be a resolving partition of $S(K_n)$. For $i, j \in [1, n]$ the representation $r(v_i|\Pi) = (0, 2, 2, \cdots, 2)$ if and only if there is no vertex v_j such that $r(v_j|\Pi) = (0, 1, 1, \cdots, 1)$.*

Proof. We assume that $r(v_i|\Pi) = (0, 2, 2, \cdots, 2)$ and $r(v_j|\Pi) = (0, 1, 1, \ldots, 1)$ for some $i, j \in [1, n]$. Since $r(v_i|\Pi) = (0, 2, 2, \cdots, 2)$, all subdivision vertices which is adjacent to v_i belong to L_1. Since $x_{i,j}$ is a subdivision vertex on v_iv_j, $x_{i,j}$ is contained in L_1. Since $r(v_j|\Pi) = (0, 1, 1, \ldots, 1)$ and $x_{i,j}$ is adjacent to v_j, we obtain $r(x_{i,j}|\Pi) = (0, 2, 2, \cdots, 2) = r(v_i|\Pi)$, a contradiction. □

Lemma 4. *If $n \geq 5$ and $\Pi = \{L_1, L_2, L_3\}$ is a resolving partition of $S(K_n)$, then the set $\{v_i|i \in [1, n]\}$ is contained in at least two partition classes in Π.*

Proof. For a contradiction, assume that $\{v_i|i \in [1, n]\} \subseteq L_1$. This implies L_2 and L_3 consist of the subdivision vertices of $S(K_n)$. So, for any $x \in L_2$ and $y \in L_3$ satisfies $d(x, y) = 2$ or $d(x, y) = 4$. By Lemma 2, we have $r(v_i|\Pi) = (0, c_2, c_3)$ where $1 \leq c_2, c_3 \leq 3$. Consider v_1 in two cases.

Case 1. v_1 is a dominant vertex. it means $r(v_1|\Pi) = (0, 1, 1)$. This implies there are at least two subdivision vertices $x_{1,2}$ and $x_{1,3}$ which are adjacent to v_1 such that $x_{1,2} \in L_2$ and $x_{1,3} \in L_3$.

Now, we consider $x_{2,3}, x_{2,4}, x_{2,5}$. Clearly, all vertices $x_{2,3}, x_{2,4}, x_{2,5} \notin L_3$ (since otherwise if one of $x_{2,3}, x_{2,4}, x_{2,5} \in L_3$ then v_2 is a dominant vertex too). If one of $\{x_{2,j}|j \in [3, 5]\} \subseteq L_2$, then $r(v_2|\Pi) = (0, 1, 3) = r(v_j|\Pi)$ (because $d(v_i, L_3) = 1$ or$d(v_i, L_3) = 3$ for each $i \in [1, n]$). This implies there are three subdivision vertices $x_{2,3}, x_{2,4}, x_{2,5}$ such that $x_{2,3}, x_{2,4}, x_{2,5} \in L_1$. On the other hand, there are only two allowed representations of these vertices, namely $(0, 2, 2)$ and $(0, 2, 4)$.

Case 2. v_1 is not a dominant vertex.
Since L_2 and L_3 consist of subdivision vertices, there is a vertex v_i such that $r(v_i|\Pi) = (0, 1, 3)$. Thus there is a subdivision vertex $x_{i,d}$ which is adjacent to v_i such that $x_{i,d} \in L_2$. Since representation $(0, 1, 3)$ is used by v_i and v_d is adjacent to $x_{i,d} \in L_2$, we obtain $d(v_d, L_3) = 1$. This implies that $r(v_d|\Pi) = (0, 1, 1)$ or v_d is a dominant vertex, which is settled in Case 1. □

Lemma 5. *Let $n \geq 5$ and $\Pi = \{L_1, L_2, L_3\}$ be a resolving partition of $S(K_n)$. Let $\{v_1, v_2, \cdots, v_n\} \subseteq L_1 \cup L_2$. If L_1 contains at least three v_i's then $r(v_i|\Pi) \neq (0, c_1, c_2)$ for all $c_1, c_2 \in [2, 3]$.*

Proof. Since $\{v_1, v_2, \cdots, v_n\} \subseteq L_1 \cup L_2$ we have $d(v_i, L_j) \leq 2$ for $j \in [1, 2]$. Let $v_1, v_2, v_3 \in L_1$ and $v_4, v_5 \in L_2$. Since L_3 does not contain a vertex v_i, by Lemma 2, we obtain $d(v_i, L_3)$ with $i \in \{1, 3\}$. This implies $(c_1, c_2) \notin \{(2, 2), (3, 3), (3, 2)\}$. To complete the proof, we will show that $(c_1, c_2) \neq (2, 3)$

Assume that there is a vertex v_i such that $r(v_i|\Pi) = (0, 2, 3)$ for $i \in [1, 3]$. Let $r(v_1|\Pi) = (0, 2, 3)$. This implies the subdivision vertices $x_{1,4}, x_{1,5} \in L_1$. Therefore, since for $i \in [1, n]$ $d(v_i, L_3) = 1$ or $d(v_i, L_3) = 3$ and $v_4, v_5 \in L_2$, we have $r(v_4|\Pi) = (1, 0, 1)$ and $r(v_5|\Pi) = (1, 0, 3)$. This implies $x_{2,4}, x_{3,4}, x_{4,5} \notin L_1$ (because if one of $x_{2,4}, x_{3,4}, x_{4,5} \in L_1$, let $x_{2,4} \in L_1$, then $r(x_{2,4}|\Pi) = r(x_{1,4}|\Pi)$). Since $r(v_4|\Pi) = (1, 0, 1)$, one of $x_{2,4}, x_{3,4}, x_{4,5}$ is in L_3. Therefore, we have exact one subdivision vertex of $x_{2,4}, x_{3,4}, x_{4,5} \in L_3$ (because if there are two $x_{2,4}, x_{3,4}, x_{4,5} \in L_3$, then both vertices' representations are equal to $(1, 1, 0)$).

Without loss of generality, let $x_{2,4} \in L_2$ and $x_{3,4} \in L_3$. Since $r(v_4|\Pi) = (1, 0, 1)$ and $x_{2,4} \in L_2$, we have $r(x_{2,4}|\Pi) = (1, 0, 2)$. Since $x_{3,4} \in L_3$, we obtain $x_{3,5} \notin L_1$ (because if $x_{3,5} \in L_1$ then $r(x_{3,5}|\Pi) = (0, 1, 2) = r(x_{1,4}|\Pi)$). Since $r(v_5|\Pi) = (1, 0, 3)$, we have $x_{3,5} \in L_2$. Therefore we obtain $r(x_{3,5}|\Pi) = (1, 0, 2) = r(x_{2,4}|\Pi)$, a contradiction. □

Corollary 6. *Let $n \geq 5$ and $\Pi = \{L_1, L_2, L_3\}$ be a resolving partition of $S(K_n)$. Let $\{v_1, v_2, \cdots, v_n\} \subseteq L_1 \cup L_2$. If L_1 contains three vertices v_1, v_2, v_3 then their representations are $(0, 1, 1), (0, 1, 3), (0, 2, 1)$.*

Proof. By Lemma 5, we have $r(v_i|\Pi) \in \{(0, 1, 1), (0, 1, 2), (0, 1, 3), (0, 2, 1)\}$ for $i \in [1, 3]$. Since L_3 only contains subdivision vertices, we obtain $d(v_i, L_3) = 1$ or $d(v_i, L_3) = 3$. This implies $r(v_i|\Pi) \in \{(0, 1, 1), (0, 1, 3), (0, 2, 1)\}$ for $i \in [1, 3]$. □

Lemma 7. *If $n \geq 5$ and $\Pi = \{L_1, L_2, L_3\}$ is a resolving partition of $S(K_n)$ then each L_k with $k \in [1, 3]$ contains v_i for some $i \in [1, n]$.*

Proof. Lemma 4 shows that all $v_i' s$ are contained in at least two partition classes of Π. Assume that $\{v_i | 1 \leq i \leq n\} \subseteq L_1 \cup L_2$. Since $n \geq 5$, one of L_1, L_2 contain at least three vertices v_i. Let L_1 contains at least three v_i. By Corollary 6, we obtain $r(v_i|\Pi) \in \{(0, 1, 3), (0, 1, 1), (0, 2, 1)\}$ for all $v_i \in L_1$. Let $r(v_1|\Pi) = (0, 1, 1)$, $r(v_2|\Pi) = (0, 1, 3)$, and $r(v_3|\Pi) = (0, 2, 1)$. Since $r(v_2|\Pi) = (0, 1, 3)$, and $r(v_3|\Pi) = (0, 2, 1)$, we have $x_{2,3} \in L_1$ and we get $r(x_{2,3}|\Pi) = (0, 2, 2)$. Therefore, we have $x_{1,2} \in L_2$ and $x_{1,3} \in L_3$ (because if $x_{1,2} \in L_1$ or $x_{1,3} \in L_1$ then $r(x_{2,3}|\Pi) = r(x_{1,2}|\Pi)$ or $r(x_{2,3}|\Pi) = r(x_{1,3}|\Pi)$). So, we get $r(x_{1,2}|\Pi) = (1, 0, 2)$.

Now, consider $x_{1,4}, x_{1,5}$. Since $r(x_{1,2}|\Pi) = (1, 0, 2)$, so we have $x_{1,4}, x_{1,5} \notin L_2$ (because if $x_{1,4} \in L_2$ or $x_{1,5} \in L_1$, then $r(x_{1,2}|\Pi) = r(x_{1,4}|\Pi)$ or $r(x_{1,2}|\Pi) = r(x_{1,4}|\Pi)$). If $x_{1,4}, x_{1,5} \in L_3$ or $x_{1,4}, x_{1,5} \in L_1$, then we obtain $r(x_{1,4}|\Pi) = r(x_{1,5}|\Pi)$. Therefore, one of $\{x_{1,4}, x_{1,5}\}$ is in L_1 and the other is in L_3. Let $x_{1,4} \in L_1$ and $x_{1,5} \in L_3$. This implies $r(x_{1,4}|\Pi) = (0, 1, 2)$ and $r(x_{1,5}|\Pi) = (1, 1, 0)$.

Next, we consider $x_{3,5}$. Since $r(v_3|\Pi) = (0, 2, 1)$, this implies we have $x_{3,5} \in L_1$ or $x_{3,5} \in L_3$. If $x_{3,5} \in L_1$, then $r(x_{3,5}|\Pi) = (0, 1, 2) = r(x_{1,4}|\Pi)$. If $x_{3,5} \in L_3$, then $r(x_{3,5}|\Pi) = (1, 1, 0) = r(x_{1,5}|\Pi)$. As consequence, each partition class L_k with $k \in [1, 3]$ must contain a vertex v_i where $i \in [1, n]$. □

Refering to Lemma 7, we obtain upper bounds for distances between vertices and partition classes in K_n which sharpen the ones in Lemma 2.

Corollary 8. *If $n \geq 5$ and $\Pi = \{L_1, L_2, L_3\}$ is a resolving partition of $S(K_n)$, then*
(i) $d(v_i, L_k) \leq 2$ for each $k \in [1, 3]$, and $i \in [1, n]$.
(ii) $d(x_{i,j}, L_k) \leq 3$ for each $k \in [1, 3]$, and $i, j \in [1, n]$.

Lemma 9. *Let $n \geq 5$ and $\Pi = \{L_1, L_2, L_3\}$ is a resolving partition of $S(K_n)$. If L_1 contains v_1, v_2, and v_3 then their representations are $(0, 1, 1), (0, 2, 1)$, and $(0, 1, 2)$.*

Proof. By Lemma 7, there are the vertices $v_i s$ in L_2 and L_3. Let $v_4 \in L_2$, $v_5 \in L_3$. By Corollary 8, the allowed representations of v_1, v_2, v_3 are $(0, 1, 1), (0, 1, 2), (0, 2, 1), (0, 2, 2)$.

We assume $r(v_1|\Pi) = (0, 2, 2)$. Hence, $x_{1,2}, x_{1,3} \in L_1$. Since $v_2, v_3 \in L_1$, we obtain $r(x_{1,2}|\Pi), r(x_{1,3}|\Pi) \in \{(0, 2, 3), (0, 3, 2)\}$. Let $r(x_{1,2}|\Pi) = (0, 2, 3)$ and $r(x_{1,3}|\Pi) = (0, 3, 2)$. This implies $r(v_2|\Pi) = (0, 1, 2)$ and $r(v_3|\Pi) = (0, 2, 1)$. Since $r(v_2|\Pi) = (0, 1, 2)$ and $r(v_3|\Pi) = (0, 2, 1)$, we have $x_{2,3} \in L_1$. So, we get $r(x_{2,3}|\Pi) = (0, 2, 2) = r(v_1|\Pi)$, a contradiction. As a consequence, the representations of v_1, v_2, v_3 are $(0, 1, 1), (0, 1, 2), (0, 2, 1)$. □

Lemma 10. *If $n \geq 5$ and $\Pi = \{L_1, L_2, L_3\}$ is a resolving partition of $S(K_n)$ then each L_k for $k \in [1,3]$ contains at most two vertices v_is.*

Proof. For a contradiction, we assume that L_1 contain three vertices v_is, i.e. v_1, v_2, v_3. By Lemma 7, suppose $v_4 \in L_2$, $v_5 \in L_3$. By Lemma 9, we have $r(v_1|\Pi) = (0,1,1)$, $r(v_2|\Pi) = (0,1,2)$ and $r(v_3|\Pi) = (0,2,1)$. This implies $r(x_{2,3}|\Pi) = (0,2,2)$.

Now, consider subdivision vertices adjacent to v_1. We obtain $x_{1,2}, x_{1,3} \notin L_1$ (since otherwise the representation of a vertex in L_1 is $(0,2,2)$ which is the same to $r(x_{2,3}|\Pi)$). Since $r(v_2|\Pi) = (0,1,2)$ and $r(v_3|\Pi) = (0,2,1)$, we obtain $x_{1,2} \in L_2$ and $x_{1,3} \in L_3$. So, we get $r(x_{1,2}|\Pi) = (1,0,2)$, $r(x_{1,3}|\Pi) = (1,2,0)$.

Consider $x_{1,4}, x_{1,5}$. If $x_{1,5} \in L_1$, then $r(x_{1,5}|\Pi) = r(v_3|\Pi)$. If $x_{1,5} \in L_3$, then $r(x_{1,5}|\Pi) = r(x_{1,3}|\Pi)$. So, we have $x_{1,5} \in L_2$. If $x_{1,4} \in L_1$, then $r(x_{1,4}|\Pi) = r(v_2|\Pi)$. If $x_{1,4} \in L_2$, then $r(x_{1,4}|\Pi) = r(x_{1,2}|\Pi)$. So, we have $x_{1,4} \in L_3$.

Next, we consider $x_{3,5}$. Since $r(v_3|\Pi) = (0,2,1)$ (it means that v_3 is not adjacent to a vertex in L_2), we obtain $x_{3,5} \in L_1$ or $x_{3,5} \in L_3$. If $x_{3,5} \in L_1$, then we have $r(x_{3,5}|\Pi) = (0,2,1) = r(v_3|\Pi)$, a contradiction. If $x_{3,5} \in L_3$, then we get $r(x_{3,5}|\Pi) = (1,2,0) = r(x_{1,2}|\Pi)$, a contradiction. As consequences, we obtain that a partition class contains at most two vertices $v_i s$. □

Lemma 10 gives a following corollary.

Corollary 11. *For $n \geq 7$, $pd(S(K_n)) \geq 4$.*

Proof. By Lemma 10, it is not possible to have only 3 partition classes for $n \geq 7$. □

Lemma 12. *Let $n \in \{5,6\}$ and $\Pi = \{L_1, L_2, L_3\}$ be a resolving partition of $S(K_n)$. Let v_i and v_j be two vertices where $i, j \in [1,n]$. If L_k contains both v_i and v_j then neither $d(v_i, L_t) = 2$ nor $d(v_j, L_t) = 2$ for $t \neq k \in [1,3]$.*

Proof. By Lemma 10, we suppose $v_1, v_2 \in L_1, v_3, v_4 \in L_2$ and $v_5 \in L_3$. For a contradiction, assume $r(v_1|\Pi) = (0,2,2)$. This implies the vertices which are adjacent to v_1, namely $x_{1,2}, x_{1,3}, x_{1,4}, x_{1,5} \in L_1$. Since $v_3, v_4 \in L_2$, we obtain $r(x_{1,3}|\Pi), r(x_{1,4}|\Pi) \in \{(0,1,2),(0,1,3)\}$. Let $r(x_{1,3}|\Pi) = (0,1,2)$ and $r(x_{1,4}|\Pi) = (0,1,3)$. So, we have $r(v_4|\Pi) = (1,0,2)$. Next, consider $x_{2,4}$. Since $r(v_4|\Pi) = (1,0,2)$, we obtain $x_{2,4} \in L_1$ or $x_{2,4} \in L_2$. If $x_{2,4} \in L_1$, then $r(x_{2,4}|\Pi) = (0,1,2) = r(x_{1,3}|\Pi)$, a contradiction. If $x_{2,4} \in L_2$, then $r(x_{2,4}|\Pi) = (1,0,3)$. Therefore, we have $r(v_2|\Pi) = (0,1,2) = r(x_{1,3}|\Pi)$, a contradiction. □

Lemma 13. *Let $n \in \{5,6\}$ and $\Pi = \{L_1, L_2, L_3\}$ be a resolving partition of $S(K_n)$. Let also $v_1, v_2 \in L_1$ and $v_3, v_4 \in L_2$. If v_1 is a dominant vertex then $x_{1,3}$ and $x_{1,4}$ belong to different partition classes of L_1 and L_2*

Proof. It is clear that $x_{1,3}, x_{1,4}$ are contained in different partition classes of Π, as otherwise $r(x_{1,3}|\Pi) = r(x_{1,4}|\Pi)$.

First, we shall show that either $x_{1,3}$ or $x_{1,4}$ is in L_1. For a contradiction, assume that both $x_{1,3}$ and $x_{1,4}$ are in $L_2 \cup L_3$. It means that $x_{1,3} \in L_2$ and $x_{1,4} \in L_3$, which implies $r(x_{1,4}|\Pi) = (1,1,0)$ and $r(x_{1,3}|\Pi) = (1,0,2)$. Since $r(x_{1,4}|\Pi) = (1,1,0)$, we have $r(v_4|\Pi) \in \{(1,0,1),(2,0,1)\}$. Now, assume $r(v_4|\Pi) = (1,0,1)$, and so, we have one of $x_{3,4}, x_{2,4}$ in L_1. If $x_{3,4} \in L_1$ then $r(x_{3,4}|\Pi) = (0,1,2) = r(v_3|\Pi)$. If $x_{2,4} \in L_1$ then $r(x_{2,4}|\Pi) = (0,1,2)$. This implies $r(v_2|\Pi) = (0,2,1)$. Therefore, we get $x_{2,3} \notin L_2$. This implies $x_{2,3} \in L_1$ (because if $x_{1,3} \in L_3$ then $r(x_{2,3}|\Pi) = (1,1,0) = r(x_{1,4}|\Pi)$). Therefore, we obtain $r(x_{2,3}|\Pi) = (0,1,2) = r(x_{2,4}|\Pi)$, a contradiction. Next, assume $r(v_4|\Pi) = (2,0,1)$. By Lemma 12 that $r(v_3|\Pi) \neq (2,0,2)$ and $r(x_{1,3}|\Pi) = (1,0,2)$, we obtain $r(v_3|\Pi) = (1,0,1)$. So, we have $x_{2,3} \in L_1$ and $x_{2,4} \notin L_3$ (because if $x_{1,3} \in L_3$ then $r(x_{1,3}|\Pi) = (1,1,0) = r(x_{1,4}|\Pi)$). Since $r(v_4|\Pi) = (2,0,1)$, we get $x_{2,4} \notin L_1$. So, we get $x_{2,4} \in L_2$. Therefore we obtain $r(x_{2,4}|\Pi) = (1,0,2) = r(x_{1,3}|\Pi)$, a contradiction As consequences of two the conditions, we obtain that one of $\{x_{1,3}, x_{1,4}\}$ is in L_1.

Without lost of generality, let $x_{1,3} \in L_1$. Lastly, we shall show that $x_{1,4}\}$ is in L_2. Assume that $x_{1,4} \in L_3$. Hence, we have $r(x_{1,3}|\Pi) = (0,1,2)$ and $r(x_{1,4}|\Pi) = (1,1,0)$. Since $(0,1,2)$ is used by $r(x_{1,3}|\Pi)$, Corollary 8 and Lemma 12, we have $r(v_2|\Pi) = (0,2,1)$. Hence, $x_{1,2} \notin L_2$. Since $r(v_1|\Pi) = (0,1,1)$, $x_{1,2} \notin L_2$, $x_{1,3} \in L_1$ and , $x_{1,3} \in L_3$, we have $x_{1,5} \in L_2$. By Corollary 8,we have $r(v_3|\Pi) = (1,0,2)$ and $r(v_4|\Pi) = (2,0,1)$. This implies that we obtan $r(x_{1,3}|\Pi) = r(x_{2,3}|\Pi)$. This completes the proof. □

Corollary 14. *Let $n \in \{5,6\}$ and $\Pi = \{L_1, L_2, L_3\}$ be a resolving partition of $S(K_n)$. If $v_1, v_2 \in L_1$, then one of v_1 or v_2 is not a dominant vertex.*

Table 1. The representations of all vertices of $S(K_8)$.

$v \in L_1$ $r(v\|\Pi')$	$v \in L_2$ $r(v\|\Pi')$	$v \in L_3$ $r(v\|\Pi')$	$v \in L_4$ $r(v\|\Pi')$
v_7 =(0,2,1,1)	v_3 =(1,0,2,1)	v_5 =(2,1,0,1)	v_1 =(2,1,2,0)
v_8 =(0,1,2,1)	v_4 =(2,0,1,1)	v_6 =(1,2,0,1)	v_2 =(2,2,2,0)
$x_{3,6}$ =(0,1,1,2)	$x_{1,3}$ =(2,0,3,1)	$x_{4,6}$ =(2,1,0,2)	$x_{1,2}$ =(3,2,3,0)
$x_{3,8}$ =(0,1,3,2)	$x_{1,4}$ =(3,0,2,1)	$x_{4,7}$ =(1,1,0,2)	$x_{1,6}$ =(2,2,1,0)
$x_{6,8}$ =(0,2,1,2)	$x_{1,5}$ =(3,0,1,1)	$x_{5,6}$ =(2,2,0,2)	$x_{1,7}$ =(1,2,2,0)
$x_{7,8}$ =(0,2,2,2)	$x_{1,8}$ =(1,0,3,1)	$x_{5,7}$ =(1,2,0,2)	$x_{2,3}$ =(2,1,3,0)
	$x_{3,4}$ =(2,0,2,2)		$x_{2,4}$ =(3,1,2,0)
	$x_{3,5}$ =(2,0,1,2)		$x_{2,5}$ =(3,2,1,0)
	$x_{4,5}$ =(3,0,1,2)		$x_{2,6}$ =(2,3,1,0)
	$x_{4,8}$ =(1,0,2,2)		$x_{2,7}$ =(1,3,2,0)
	$x_{5,8}$ =(1,0,1,2)		$x_{2,8}$ =(1,2,3,0)
			$x_{3,7}$ =(1,1,2,0)
			$x_{6,7}$ =(1,3,1,0)

Proof. By Lemma 10, we suppose $v_1, v_2 \in L_1, v_3, v_4 \in L_2$ and $v_5 \in L_3$. For a contradiction, assume $r(v_1|\Pi) = (0, 1, 1)$. By Lemma 13, we obtain $x_{1,3} \in L_1$ and $x_{1,4} \in L_2$. So, we have $r(x_{1,3}|\Pi) = (0, 1, 2)$ and $r(x_{1,4}|\Pi) = (1, 0, 2)$. Hence, we obtain $r(v_3|\Pi) = (1, 0, 1)$. Therefore, by Lemma 12, we get $r(v_4|\Pi) = (2, 0, 1)$.

Next, since $r(x_{1,3}|\Pi) = (0, 1, 2)$, $r(v_1|\Pi) = (0, 1, 1)$ and by Lemma 12, we have $r(v_2|\Pi) = (0, 2, 1)$. Since $r(v_4|\Pi) = (2, 0, 1)$ and $r(v_2|\Pi) = (0, 2, 1)$, we obtain $x_{2,4} \in L_3$. Hence $x_{2,3} \notin L_3$ (because if $x_{2,3} \in L_3$ then $r(x_{2,3}|\Pi) = r(x_{2,4}|\Pi)$). Since $r(v_2|\Pi) = (0, 2, 1)$, this means that V_2 is not adjacent to a vertex in L_2. So we have $x_{2,4} \in L_1$. This implies $r(x_{2,3}|\Pi) = (0, 1, 2) = r(x_{1,3}|\Pi)$, a contradiction. □

Let G be a connected graph and $v \in V(K_n)$. The *open neighbourhood* of v, $N(v) = \{x \in V(G)|vx \in E(G)\}$ and the *closed neighbourhood* of v, $N[v] = N(v) \cup \{v\}$.

Theorem 15. $pd(S(K_n)) = \begin{cases} 2 \text{ if } n = 2, \\ 3 \text{ if } n \in [3, 4] \\ 4 \text{ if } n \in [5, 8]. \end{cases}$

Proof. For $n = 2$, K_2 is a path, and so the graph $S(K_n)$ is also a path. This implies $pd(S(K_n))) = 2$. For $n = 3, 4$, we obtain that $S(K_n)$ is not a path. Therefore, $pd(S(K_n)) \geq 3$. Let $\Pi = \{L_1, L_2, L_3\}$ be a partition of $V(S(K_n))$ as depicted in Figure 1. It is easy to verify that Π is a resolving partition of $S(K_n)$.

For $n = 5, 6$, by a contradiction, we assume $\Pi = \{L_1, L_2, L_3\}$ is a resolving partition of $S(K_n)$. Since $n = 5, 6$ and by Lemma 10, we have that there exits at most two v_i which are in the a partition class L_i. Let $v_1, v_2 \in L_1$, $v_3, v_4 \in L_2$ and $v_5 \in L_3$. By Corollary 14 and Lemma 12, we obtain $r(v_1|\Pi) = (0, 1, 2)$, $r(v_2|\Pi) = (0, 2, 1)$ and $r(v_3|\Pi) = (1, 0, 2)$, $r(v_4|\Pi) = (2, 0, 1)$. Since $r(v_2|\Pi) = (0, 2, 1)$ and $r(v_3|\Pi) = (1, 0, 2)$, we get $x_{2,4} \in L_1$. So, we obtain $r(x_{2,4}|\Pi) = (0, 1, 2) = r(v_1|\Pi)$, a contradiction. Therefore, we have $pd(S(K_n)) \geq 4$.

To show $pd(S(K_8)) \leq 4$, define a partition $\Pi' = \{L'_1, L'_2, L'_3, L'_4\}$ of $V(S(K_8))$, see Figure 2, where
$L'_1 = \{v_8, v_7, x_{7,8}, x_{6,8}, x_{3,8}, x_{3,6}\}$,
$L'_2 = \{v_3, v_4, x_{3,4}, x_{1,4}, x_{1,5}, x_{4,8}, x_{4,5}, x_{3,5}, x_{5,8}, x_{1,8}, x_{1,3}\}$,
$L'_3 = \{v_5, v_6, x_{5,6}, x_{4,7}, x_{4,6}, x_{5,7}\}$ and
$L'_4 = \{v_1, v_2, x_{1,2}, x_{2,3}, x_{2,8}, x_{2,7}, x_{2,6}, x_{2,5}, x_{2,4}, x_{6,7}, x_{1,7}, x_{1,6}, x_{3,7}\}$.
The representations of all vertices are shown in Table 1. It is easy to verify that Π' is a resolving partition of $S(K_8)$.

For $S(K_7) = S(K_8) - N[v_3]$, all subdivision vertices which are adjacent to v_3, namely $x_{1,3}, x_{2,3}, x_{3,4}, \cdots, x_{3,8}$. Now, let $a_i = min\{i, 3\}$, $b_i = max\{i, 3\}$. Since for $i \in \{2, 4, 8\}$ each x_{a_i,b_i} is contained in the same partition class containing v_i, deleting these vertices in $S(K_8)$ do not change $r(v_i|\Pi'.)$ Meanwhile for $i \in \{1, 5, 6, 7\}$, each v_i is contained in distinct partition class with x_{a_i,b_i} and it is adjacent to other subdivision vertex which lies in the same partition class with x_{a_i,b_i}. So, deleting each x_{a_i,b_i} in $S(K_8)$ doesn't change $r(v_i|\Pi')$ for $i \in \{1, 5, 6, 7\}$. Since L_2 contains two vertices v_2 and v_3, removing $N[v_3]$ in $S(K_8)$ do not change the representations of all the remaining vertices. Hence, $\Pi'' = \{L''_1, L''_2, L''_3, L''_4\}$ is a resolving partition of $S(K_7)$ where $L''_i = L'_i - \{x|x \in L'_i \cap N[v_3]\}$. Since deleting $N[v_3]$ do not change all remaining vertices in $S(K_8)$, we have Π'' as a resolving partition of $S(K_7)$. Therefore, we obtain $pd(S(K_7)) = 4$.

We can see that $N[v_5]$ and $N[v_7]$ has the similar property as $N[v_3]$. Therefore by similar way, we have $pd(S(K_5)) = pd(S(K_6)) = 4$ where $S(K_6) = S(K_7) - N[v_5]$ and $S(K_5) = S(K_6) - N[7]$. □

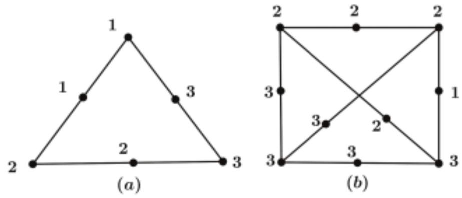

Fig. 1. A resolving partition of $S(K_3)$ and $S(K_4)$.

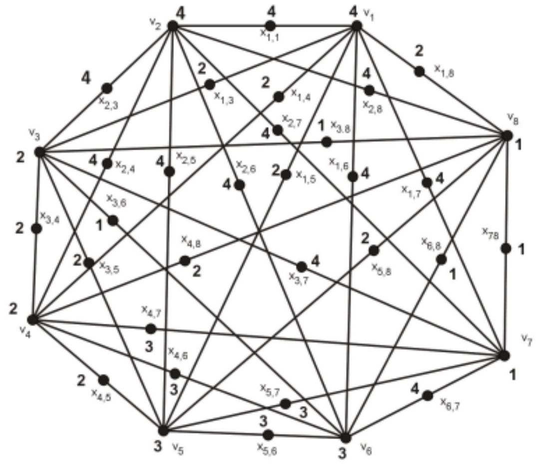

Fig. 2. A resolving partition of $S(K_8)$.

Lemma 16. *Let $\Pi = \{L_1, L_2, \cdots, L_p\}$ be a resolving partition of $S(K_n)$ and each L_j contains v_k for some $k \in [1, n]$. If there is a partition class L_c such that $d(v_i, L_c) \leq 1$ for all $i \in [1, n]$, then $pd(S(K_{n+1})) \leq p+1$ and $pd(S(K_{n+2})) \leq p+1$.*

Proof. The proof is divided into two parts:
First, we will show $pd(S(K_{n+1})) \leq p + 1$. Let $\Pi' = \{L'_1, L'_2, \cdots, L'_p, L'_{p+1}\}$ be a partition of $V(S(K_{n+1}))$ where $L'_c = L_c \cup \{x_{1,n+1}, x_{2,n+1}, \cdots, x_{n,n+1}\}$, $L'_i = L_i$ for $i \in [1, p]$, $i \neq c$, and $L'_{p+1} = \{v_{n+1}\}$. We have to note that L'_c satisfies $d(v_i, L'_c) \leq 1$ for all $i \in [1, n+1]$ also. This fact can be used to construct a resolving partition of $S(K_{n+2})$.

Let $G' = K_{n+1}$, $B = N[v_{n+1}]$, and $C = V(S(G')) \setminus B$. Let u, w be two distinct vertices in the same partition class of Π'. Since $L'_{p+1} = \{v_{n+1}\}$, $\{x_{1,n+1}, x_{2,n+1}, \cdots, x_{n,n+1}\} \subseteq L'_c$, and $L'_i = L_i$ for $i \in [1, p], i \neq c$, we obtain $d(v, L'_i) = d(v, L_i)$ for all $v \in C$. Hence, for all $u, w \in C$, if u, w are distinguished by L_t in Π with $t \in [1, p]$, then u, w are distinguished by L'_t di Π'.

Next, we consider $u \in B$. It means that $u = x_{i,n+1} \in L'_c$ or $u = v_{n+1} \in L'_{p+1}$ for $i \in [1, n]$. We will show that the vertex u has distinct representation with the other vertex w in $V(S(G'))$. For $u \in L'_{p+1}$, it has distinct representation with the others in $S(G')$ because L'_{p+1} only contain one vertex. If $u = x_{i,n+1}$ and $w \in L'_c - B$ with $i \in [1, n]$, then u, v are distinguished by L'_{p+1} (because $d(u, L'_{p+1}) = 2$ and $d(w, L'_{p+1}) = 1$). If $u = x_{i,n+1}$ and $w = x_{j,n+1}$ for $i \neq j \in [1, n]$, then consider v_i and v_j. We know that v_i is adjacent to u and v_j is adjacent to w. If v_i, v_j are in the same partition class, then there is a partition class L_d in Π for some $d \in [1, p]$ such that v_i, v_j are distinguished by L_d in Π for $i \in [1, n]$. Since each L_k contains a vertex v_i, and the vertices v_i, v_j are distinguished by L_d in Π, we obtain u, w are distinguished by L'_d in $S(G')$. If v_i, v_j are in the different partition classes, then u, w are distinguished by v_i or v_j in $S(G')$. As consequences,

Π' is a resolving partition of $S(G')$. Therefore, we have $pd(S(G')) \leq p+1$.

Second, let $G'' = K_{n+2}$, $B = N[v_{n+2}]$, and $C = V(S(G''))\setminus B$. We will show $pd(G'') \leq p+1$. Let $\Pi'' = \{L''_1, L''_2, \cdots, L''_p, L''_{p+1}\}$ be a partition of $V(G'')$ where $L''_i = L'_i$ for $i \in [1,p]$ and $L''_{p+1} = L'_{p+1} \cup N[v_{n+2}]$. We note that L''_{p+1} satisfies $d(v_i, L''_{p+1}) \leq 1$ for all $i \in [1, n+2]$. So, the fact can be used to construct a resolving partition of $S(K_{n+3})$.

Since $L''_i = L'_i$ and $L''_{p+1} = L'_{p+1} \cup N[v_{n+2}$, we obtain that any $u \in C$ have $d(u, L''_k) = d(u, L'_k)$ for $k \in [1,p]$. Hence, for two distinct vertices u, w in L''_j where $j \in [1,p]$ we have $r(u|\Pi'') \neq r(w|\Pi'')$.

Next, consider $u, w \in L_{p+1}$. If $u = v_{n+1}$ and $w = v_{n+2}$, then u, w are distinguished by L''_c, (because $d(u, L''_c) = 1$ and $d(w, L''_c) = 2$). If $u = x_{i,n+2}$ and $w = x_{j,n+2}$ where $i \neq j \in [1, n+1]$, then consider v_i and v_j. We can see that v_i is adjacent to u and v_j is adjacent to w for $i, j \in [1, n+1]$. If v_i, v_j are in the same partition class, then there is a partition class L'_d in Π' such that $v_i v_j$ are distinguished by L'_d in Π'. Since each L'_k contains a vertex v_i for $i \in [1, n]$, the vertices $v_i v_j$ are distinguished by L'_d, and v_{n+1} is only adjacent to vertices in L'_c, we obtain u, w are distinguished by L''_d in $S(G'')$. If v_i, v_j are in the different partition classes, then u, w are distinguished by v_i or v_j in $S(G'')$.

If $u = v_{n+2}$ and $w = x_{i,n+2}$, then $r(u|\Pi'')$ has not a component which is value '1' and $r(u|\Pi'')$ has a component which is value '1'. So, we have $r(u|\Pi'') \neq r(w|\Pi'')$. If $u = v_{n+1}$ and $w = x_{i,n+2}$, then we consider v_i which is adjacent to w for some $i \in [1, n+1]$. If $v_i \notin L''_p$ then u, v are distinguished by L''_p (because u is only adjacent to vertices in $L''_p \cup L''_{p+1}$). If $v_i \in L''_p$, then u, w are distinguished by L''_t where L'_t is a partition class distinguishing v_i, $x_{j,n+1}$ with $j \in [1, n]$ in $S(K_{n+1})$. Hence, we have $r(u|\Pi'') \neq r(w|\Pi'')$. As consequences, Π'' is a resolving partition of $S(G'')$, so $pd(S(G'')) \leq p+1$. □

Theorem 17. *If $n \geq 9$, then $pd(S(K_n)) \leq \lceil \frac{n}{2} \rceil$.*

Proof. Consider $S(K_8)$ with $\Pi = \{L_1, L_2, L_3, L_4\}$ be a partition of $V(S(K_n))$ with L_i=L'_i where L'_i is the partition class of Π' on the Theorem 15. We can see that Π satisfies the condition in Lemma 16. Furthermore, the partition class L_4 satisfies $d(v_i, L_4) \leq 1$ for all $i \in [1,8]$, Hence, by the constructions in Lemma 16, we obtain $pd(S(K_9)) \leq 5$ and $pd(S(K_{10})) \leq 5$. Now, repeat the same process recursively to obtain $pd(S(K_n)) \leq \lceil \frac{n}{2} \rceil$ for $n \geq 9$. □

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