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The Degree, Radius, and Diameter of Coprime Graph of Dihedral Group

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Abstract. The coprime graph Γ_G with a finite group G define as a collection of vertex set G with two distinct vertices x and y if only if $(|x|,|y|) = 1$. In 2019, Gazir. at all found the form of the coprime graph of the dihedral group is complete tripartite graph and complete bipartite graph. In this paper we found that if $\Gamma_{D_{2n}}$ complete bipartite graph then the radius is 1, the diameter is 2 and and the degree of elements are $2n - 1$ and 1.

Keywords: coprime graph, dihedral group, radius, diameter, degree

1. Iintroduction

Graph theory is a subject that had emerged long ago, but has many applications to the present. Graph is used to present discrete objects and the relationships between them. In recent years, many different types of graph have existed from a group defined by mathematicians from various parts of the world. Some types of those graphs are commuting and non-commuting graphs, cyclic graph, identity graph, division by zero graph and other graphs of a mathematical system.

In 2014, X. MA, et al. defined a new graph named Coprime graph. As example, if $\boldsymbol{G}$ is finite group, the coprime graph of group $\boldsymbol{G}$ denoted by $\boldsymbol{\Gamma_G}$ is a graph with vertices consisting of all elements of $\boldsymbol{G}$ with two different vertices said to be adjency if only if they have a prime order. Furthermore, in 2015, Rajkumar and Devi analyzed several properties of the coprime graph from a subgroup. In 2017, Abdussakir discussed about the commuting graphs of the dihedral group, so that there were several properties regarding the radius, diameter, multiplicity cycel and metric dimensions of the graph are obtained. In 2019, Gazir et al. pointed out one of the coprime graph properties which stated that the shape of the coprime graph of the dihedral group is a multipartite graph if n are prime number.

2. Result

This section will discuss the dihedral group and its representation towards the coprime graph and the graph properties.

2.1. Definition (Dummit dan Foote, 2004)
Suppose $\boldsymbol{G}$ is a finite group. Group $\boldsymbol{G}$ which called a dihedral group with order $\boldsymbol{2n}, \boldsymbol{n} \geq \boldsymbol{3}$, is a group genereted by two elements a, b with properties

$$\boldsymbol{G = \langle a, b | a^n = e, b^2 = e, bab^{-1} = a^{-1} \rangle}$$

The dehidral group with order $\boldsymbol{2n}$ is denoted by $\boldsymbol{D_{2n}}$. From the definition it is easy to see that $|\boldsymbol{D_{2n}}| = \boldsymbol{2n}$ and can be written as a set with
$\boldsymbol{D_{2n}} = \{\boldsymbol{e, a, a^2, a^3}, \dots, \boldsymbol{a^{n-1}, b, ab, a^2b, a^3b}, \dots, \boldsymbol{a^{n-1}b}\}$.

Coprime graph is a new way to present a group, introduced in 2014 by X Ma et al. Coprime graph involve the order of group elements. In this study, Coprime graphs will be used to represent dihedral group. Some examples of coprime graphs from $\boldsymbol{D_6}$ to $\boldsymbol{D_{10}}$ with $\boldsymbol{n = 3, 4, 5}$.
Coprime graph of dihedral group D_6

Table 2.1 Order of dihedral group D_6

Elements	e	a	a^2	b	ab	a^2b
Order	1	3	3	2	2	2

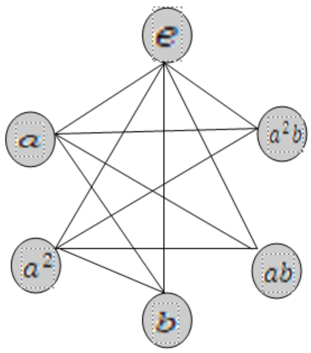

Figure 2.1 Coprime graph of dihedral group D_6

Based on the graph in the figure obtained that:

a. Form of coprime graph of dihedral group D_6 is complate tripartite
b. Radius of coprime graph of dihedral group D_6 is 1, yaitu dari nilai $e(1)$.
c. Diameter of coprime graph of dihedral group D_6 is 2
d. Degree of vertices of coprime graph of dihedral group D_6 is 5, 4, and 3.

Coprime graph of dihedral group D_8

Tabel 2.2 Order of dihedral group D_8

Elements	e	a	a^2	a^3	b	ab	a^2b	a^3b
Order	1	4	2	4	2	2	2	2

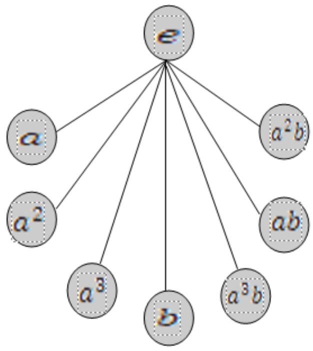

Figure 4.2 Coprime graph of dihedral group D_8

Based on the graph in the figure obtained that::

a. Form of coprime graph of dihedral group D_8 is complate bipartite
b. Radius of coprime graph of dihedral group D_8 is 1, value of $e(1)$.
c. Diameter of coprime graph of dihedral group D_8 is 2
d. Degree of vertices of coprime graph of dihedral group D_8 is 7 and 1

Coprime graph of dihedral group D_{10}

Tabel 2.3 Order of dihedral group D_{10}

Elements	e	a	a^2	a^3	a^4	b	ab	a^2b	a^3b	a^4b
Order	1	5	5	5	5	5	5	5	5	5

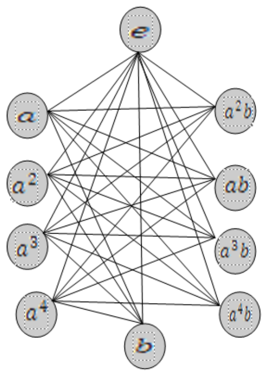

Figure 4.3 Coprime graph of dihedral group D_{10}

Based on the graph in the figure obtained that::

a. Form of coprime graph of dihedral group D_{10} is complate tripartite.
b. Radius of coprime graph of dihedral group D_{10} is 1, value of $e(1)$.
c. Diameter of coprime graph of dihedral group D_{10} is 2.
d. Degree of vertices of coprime graph of dihedral group D_{10} is 9, 6, and 5.

In the same way for each n a number of patterns are obtained which are then outlined in the following theorems:

The coprima graph form of D_{2n}, where n prime numbers are explained in the following theorem.

***2.2. Theorem** (Gazir et al., 2019)*
Let $\boldsymbol{D_{2n}}$ a dihedral group where $\boldsymbol{n \geq 3}$. If *n* is prime number, the coprime graph of $\boldsymbol{D_{2n}}$ is a complete tripartite graph.
Proof.

Suppose $\boldsymbol{D_{2n}}$ is dihedral group with n prime number, then $\boldsymbol{D_{2p} = \{e, a, a^2, \dots, a^{p-1}, b, ab, \dots, a^{p-1}b\}}$ then the dihedral group partition derives as 3 partitions as $\boldsymbol{V_1 = \{e\}}$, $\boldsymbol{V_2 = \{a, a^2, \dots, a^{p-1}\}}$ and $\boldsymbol{V_3 = \{\, b, ab, a^2b, \dots, a^{p-1}b\}}$. It is easy to see $|\boldsymbol{e}|$=1, $|\boldsymbol{a}| = |\boldsymbol{a^2}| = \cdots |\boldsymbol{a^{p-1}}| = \boldsymbol{p}$ because *n* is prime number$|\boldsymbol{b}| = |\boldsymbol{ab}| = |\boldsymbol{a^2b}| = \cdots = |\boldsymbol{a^{p-1}b}| = \boldsymbol{2}$ is the result of reflexive property. For each $\boldsymbol{x, y} \in \boldsymbol{V_2}$, then $(|\boldsymbol{x}|, |\boldsymbol{y}|) = \boldsymbol{n \neq 1}$, as well as for every $\boldsymbol{r, s} \in \boldsymbol{V_3}$ then $(|\boldsymbol{r}|, |\boldsymbol{s}|) = \boldsymbol{2 \neq 1}$. For any $\boldsymbol{u} \in \boldsymbol{V_i}$ and $\boldsymbol{v} \in \boldsymbol{V_j}$when $\boldsymbol{i \neq j}$ then $(|\boldsymbol{u}|, |\boldsymbol{v}|) = \boldsymbol{1}$. Thus, $\boldsymbol{u}$ and $\boldsymbol{v}$ are adjacent so that the coprime graph form of the dihedral group is a complete tripartite. ■

In addition to forming a complete tripartite graph, the coprima graph of D_{2n} is a complete bipartite graph when $n = 2^k$, for a $k \in N$.

***2.3. Theorem** (Gazir et al., 2019)*
Let D_{2n} a dihedral group where $n \geq 3$, if $n = 2^k$, for a $k \in N$ then the coprime graph of D_{2n} is a complete bipartite graph.
Proof.
Suppose D_{2n} is a dihedral group with $n = 2^k$ then $\boldsymbol{D_{2.2^k} = \{e, a, a^2, \dots, a^{2^k-1}, b, ab, \dots, a^{2^k-1}b\}}$ then the dihedral group partition derives as 2 partitions as $V_1 = \{e\}$, and $V_2 = \{a, a^2, \dots, a^{2^k-1}, b, ab, a^2b, \dots, a^{2^k-1}b\}$. It is easy to see $|e|$=1, $|a| = |a^2| = \cdots |a^{2^k-1}| = 2^s$ for a $s \in N$ because of $n = 2^k$, $|b| = |ab| = |a^2b| = \cdots = |a^{p-1}b| = 2$ is the result of reflexive property. For each $x, y \in V_2$then $(|x|, |y|) = 2 \neq 1$, thus x and y are not neighbors. Because of $|e|$=1 and for any $v \in V_2$ then $(|e|, |v|) = 1$. So, e is adjacent to every other vertex then that the form of the coprime graph of the dihedral group is complete bipartite. ■

The following theorem explains that the coprima graph of D_{2n} is a complete tripartite graph as well as typing prime numbers.

***2.4. Theorem** (Gazir et al., 2019)*
Let D_{2n} a dihedral group where $n \geq 3$. If $n = p^k, p \neq 2$, for $k \in N$ then the coprime graph of D_{2n} is a complete tripartite graph.
Proof.

Suppose that group $\boldsymbol{D_{2n}}$ is dihedral with n prime number, then $\boldsymbol{D_{2p^k} = \{e, a, a^2, \dots, a^{p^k-1}, b, ab, \dots, a^{p^k-1}b\}}$ then the dihedral group partition derives as 3 partitions as $\boldsymbol{V_1 = \{e\}}$, $\boldsymbol{V_2 = \{a, a^2, \dots, a^{p^k-1}\}}$ and $\boldsymbol{V_3 = \{\, b, ab, a^2b, \dots, a^{p^k-1}b\}}$. It is easy to see $|\boldsymbol{e}|$=1, $|\boldsymbol{a}| = |\boldsymbol{a^2}| = \cdots |\boldsymbol{a^{p^k-1}}| = \boldsymbol{p^s}$ for any $\boldsymbol{s \in N}$ because of $\boldsymbol{n = p^k, p \neq 2}$, **for any** $\boldsymbol{k \in N}$ and $|\boldsymbol{b}| = |\boldsymbol{ab}| = |\boldsymbol{a^2b}| = \cdots = |\boldsymbol{a^{p^k-1}b}| = \boldsymbol{2}$ is the result of reflexive property. For each $\boldsymbol{x, y} \in \boldsymbol{V_2}$, then $(|\boldsymbol{x}|, |\boldsymbol{y}|) = \boldsymbol{n \neq 1}$, as well as for every $\boldsymbol{r, s} \in \boldsymbol{V_3}$ then $(|\boldsymbol{r}|, |\boldsymbol{s}|) = \boldsymbol{2 \neq 1}$. For any $\boldsymbol{u} \in \boldsymbol{V_i}$ and $\boldsymbol{v} \in \boldsymbol{V_j}$

when $\boldsymbol{i} \neq \boldsymbol{j}$ then $(|\boldsymbol{u}|, |\boldsymbol{v}|) = \mathbf{1}$. Thus, $\boldsymbol{u}$ and $\boldsymbol{v}$ are adjacent then that the coprime graph form of the dihedral group is a complete tripartite. ■

The following several theorems explain other characteristics of the coprima graph, namely the degree of vertices. In the following theorem, it will be explained that typing prime numbers has three different degree values.

2.5. Theorem
Let $\boldsymbol{D_{2n}}$ a dihedral group where $\boldsymbol{n} \geq \mathbf{3}$. If n is a prime number, then the degree of vertices in coprime graph of $\boldsymbol{D_{2n}}$ are
a. $\mathbf{deg}_{\boldsymbol{D_{2n}}}(\boldsymbol{a^i}) = \boldsymbol{n} + \mathbf{1}$ for every $\boldsymbol{i} \in \boldsymbol{Z}, \mathbf{1} \leq \boldsymbol{i} < \boldsymbol{n}$
b. $\mathbf{deg}_{\boldsymbol{D_{2n}}}(\boldsymbol{a^i b}) = \boldsymbol{n}$ for every $\boldsymbol{i} \in \boldsymbol{Z}, \mathbf{1} \leq \boldsymbol{i} < \boldsymbol{n}$
c. $\mathbf{deg}_{\boldsymbol{D_{2n}}}(\boldsymbol{e}) = \mathbf{2}\boldsymbol{n} - \mathbf{1}$
Proof.

Suppose $\boldsymbol{D_{2n}}$ is a dihedral group with n prime number, then $\boldsymbol{D_{2p}} = \{\boldsymbol{e}, \boldsymbol{a}, \boldsymbol{a^2}, \dots, \boldsymbol{a^{p-1}}, \boldsymbol{b}, \boldsymbol{ab}, \dots, \boldsymbol{a^{p-1}b}\}$. Based on theorem 2.2 it is known that the form of the coprime graph of $\boldsymbol{D_{2n}}$ is a complete tripartite graph with partition $\boldsymbol{V_1} = \{\boldsymbol{a^i}\}$ with a number of members as $\boldsymbol{n} - \mathbf{1}$, $\boldsymbol{V_2} = \{\boldsymbol{a^i b}\}$ with a number of members as n and $\boldsymbol{V_3} = \{\boldsymbol{e}\}$ with only 1 member. It's easy to see that vertex $\boldsymbol{e}$ is connected to every other vertex which means $(\boldsymbol{e}) = \mathbf{2}\boldsymbol{n} - \mathbf{1}$. Then each rotation element vertex is connected to the vertex $\boldsymbol{e}$ and all reflection element vertices are $(\boldsymbol{a^i}) = \boldsymbol{n} + \mathbf{1}$ for each $\boldsymbol{i} \in \boldsymbol{Z}, \mathbf{1} \leq \boldsymbol{i} < \boldsymbol{n}$. Likewise, each vertex of the reflection element is connected to each vertex of rotation element, so that $(\boldsymbol{a^i}\ \boldsymbol{b}) = \boldsymbol{n}$ for every $\boldsymbol{i} \in \boldsymbol{Z}, \mathbf{1} \leq \boldsymbol{i} < \boldsymbol{n}$. ■

In line with the form of the coprima graph of the group which is partitioned into two and forms a complete bipartite graph when $n = 2^k$, for a $k \in N$, the following theorem explains that the degree value of vertices has two different degree values.

2.6. Theorem
Let $\boldsymbol{D_{2n}}$ a dihedral group where $\boldsymbol{n} \geq \mathbf{3}$. If $\boldsymbol{n} = \mathbf{2^k}$, for a $\boldsymbol{k} \in \boldsymbol{N}$, then the degree of vertices in coprime graph of $\boldsymbol{D_{2n}}$ are
a. $\deg_{D_{2n}}(a^i) = \deg_{D_{2n}}(a^i b) = 1$ for every $i \in Z, 1 \leq i < n$
b. $\deg_{D_{2n}}(e) = 2n - 1$
Proof.

Suppose $\boldsymbol{D_{2n}}$ is a dihedral group with $\boldsymbol{n} = \mathbf{2^k}$, for $\boldsymbol{k} \in \boldsymbol{N}$, then $\boldsymbol{D_{2.2^k}} = \{\boldsymbol{e}, \boldsymbol{a}, \boldsymbol{a^2}, \dots, \boldsymbol{a^{2^k-1}}, \boldsymbol{b}, \boldsymbol{ab}, \dots, \boldsymbol{a^{2^k-1}b}\}$. Based on theorem 2.3 it is known that the form of the coprime graph of $\boldsymbol{D_{2n}}$ is a complete bipartite. It's easy to see that vertex $\boldsymbol{e}$ is connected to every other vertex which means $(\boldsymbol{e}) = \mathbf{2}\boldsymbol{n} - \mathbf{1}$. Each rotation and reflection element vertex is connected to vertex $\boldsymbol{e}$ then $\mathbf{deg}_{\boldsymbol{D_{2n}}}(\boldsymbol{a^i}) = \mathbf{deg}_{\boldsymbol{D_{2n}}}(\boldsymbol{a^i b}) = \mathbf{1}$ for every $\boldsymbol{i} \in \boldsymbol{Z}, \mathbf{1} \leq \boldsymbol{i} < \boldsymbol{n}$. ■

The following explains that when $n = p^k$, $p \neq 2$, for a $k \in N$ then the degree of verticeshas three different values of the degree of vertices.

2.7. Theorem
Let $\boldsymbol{D_{2n}}$ a dihedral group where $\boldsymbol{n \geq 3}$. If $\boldsymbol{n = p^k, p \neq 2}$, for a $\boldsymbol{k} \in \boldsymbol{N}$, then the degree of vertices in coprime graph of $\boldsymbol{D_{2n}}$ are

a. $\mathbf{deg}_{\boldsymbol{D_{2n}}}(\boldsymbol{a^i}) = \boldsymbol{n+1}$ for every $\boldsymbol{i} \in \boldsymbol{Z}, \boldsymbol{1 \leq i < n}$
b. $\mathbf{deg}_{\boldsymbol{D_{2n}}}(\boldsymbol{a^i b}) = \boldsymbol{n}$ for every $\boldsymbol{i} \in \boldsymbol{Z}, \boldsymbol{1 \leq i < n}$
c. $\mathbf{deg}_{\boldsymbol{D_{2n}}}(\boldsymbol{e}) = \boldsymbol{2n-1}$

Proof.

Suppose D_{2n} is a dihedral group with $n = p^k$, $p \neq 2$ for $k \in N$, then $D_{2.p^k} = \{e, a, a^2, \dots, a^{p^k-1}, b, ab, \dots, a^{p^k-1}b\}$. Based on theorem 2.4 it is known that the form of the coprime graph of D_{2n} is a complete tripartite. Then from theorem 2.5 it is known $\deg_{D_{2n}}(\boldsymbol{a^i}) = \boldsymbol{n} + \boldsymbol{1}$ for every $i \in Z, 1 \leq i < n$, $\deg_{D_{2n}}(a^i b) = n$ for every $i \in Z, 1 \leq i < n$, and $\deg_{D_{2n}}(e) = 2n - 1$. ■

The following theorem explains the characteristics of other coprima graphs, namely radius and diameter. For any n, the values of radius and diameter are one and two, respectively.

2.8. Theorem
Suppose that $\boldsymbol{\Gamma_{D_{2n}}}$ is a coprime graph of dihedral group when $\boldsymbol{n \geq 3}$, then the radius and diameter of $\boldsymbol{\Gamma_{D_{2n}}}$ for each are ($\boldsymbol{D_{2n}}$) are $\boldsymbol{rad(\Gamma_{D_{2n}}) = 1}$ dan $\boldsymbol{diam(\Gamma_{D_{2n}}) = 2}$.
Proof.

It is known that $\boldsymbol{gcd}\ (|\boldsymbol{e}|, |\boldsymbol{x}|) = \boldsymbol{1}$ is for all $\boldsymbol{x} \in \boldsymbol{D_{2n}}$. Thus vertex $\boldsymbol{e}$ will be connected directly to all other vertices in $\boldsymbol{D_{2n}}$ as a result e (e) = 1, because the radius of $\boldsymbol{D_{2n}}$ is the smallest eccentricity in $\boldsymbol{D_{2n}}$, then $\boldsymbol{rad(\Gamma_{D_{2n}}) = 1}$ is obtained. Since all vertices are directly connected to vertex $\boldsymbol{e}$, the distance of two different vertices in $\boldsymbol{D_{2n}}$ has only two possibilities, namely 1 or 2. Two vertices will be spaced 1 if they are directly connected to each other and be spaced 2 if they are not directly connected because $\boldsymbol{gcd}\ (|\boldsymbol{a}|, |\boldsymbol{b}|) \neq \boldsymbol{1}$, when $\boldsymbol{a, b} \in \boldsymbol{D_{2n}}$ so that $\boldsymbol{d(a, b) = 2}$. Thus it is obtained $\boldsymbol{e(a) = e(b) = 2}$, because the diameter is the greatest eccentricity, then $\boldsymbol{diam(\Gamma_{D_{2n}}) = 2}$. ■

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